$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

$$\Phi = \frac{U(r)}{r}P(\theta)\,Q(\phi)$$

$$Q_m = e^{\pm im\phi}$$

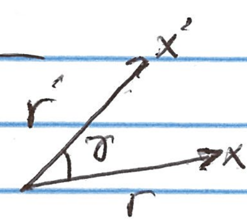
$$U_\ell = Ar^{\ell+1} + Br^{-\ell}$$

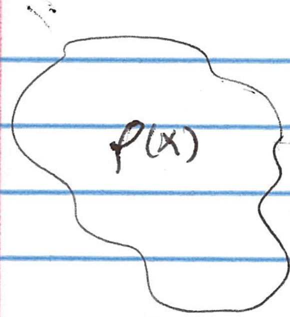
P_ℓ = Legendre Polynomial

Azimutal Symetry

$$\boxed{\Phi(r,\theta) = \sum_{\ell=0}^{\infty}\left[A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right]P_\ell(\cos\theta)}$$

$$\frac{1}{|x-x'|} = \sum_{\ell=0}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}}P_\ell(\cos\gamma)$$





$$\boxed{\Phi(x) = \frac{1}{4\pi\varepsilon_0}\int dx'\,\rho(x')\sum_{\ell=0}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}}P_\ell(\cos\gamma)}$$

$$\frac{1}{|x-x'|} = \sum_{\ell=1}^{\infty}\frac{1}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}Y_{\ell m}^*(\theta',\phi')Y_{\ell m}(\theta,\phi)$$

$$\Phi(\rho,\phi,z) = \sum \left\{ J_m(k\rho) \text{ or } Y_m(k\rho) \right\} \left\{ e^{\pm im\phi} \right\} \left\{ e^{\pm kz} \right\}$$

if $k \longrightarrow ik$ then we have $I_m(k\rho)$ & $K_m(k\rho)$

modified Bessel function.

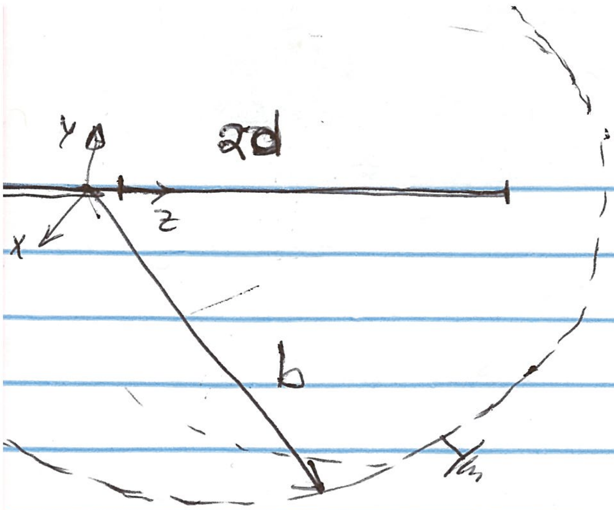
Green Function

$$\boxed{\begin{aligned} &G(x,x') = \frac{1}{|x-x'|} + f(x,x') \\ &\nabla^2 f(x,x') \end{aligned}}$$

$$\frac{1}{|x-x'|} = \sum_{\ell} \frac{r_<^{\ell}}{r_>^{\ell+1}} P_\ell(\cos\theta)$$

for example

$$f(r,r') = A_\ell r'^{\ell} + \frac{B_\ell}{r'^{\ell+1}}$$



total charge: Q

charge density: $(d^2 - z^2)$

$$\boxed{Q = \int_{-d}^{d} K(d^2 - z^2)\,dz = \frac{4}{3} K d^3} \longrightarrow K = \frac{3Q}{4d^3}$$

$$\boxed{\rho \propto \frac{3Q}{4d^3}(d^3 - z^3)} \qquad \rho(r,\theta,\phi) = \frac{3Q}{4d^3}(d^2 - r^2)\frac{1}{\pi r^2}\delta(\cos^2\theta - 1)$$

$$G(x,x') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y^*_{lm}(\theta',\phi') Y_{lm}(\theta,\phi)}{(2l+1)\left[1 - \left(\frac{a}{b}\right)^{2l+1}\right]} \left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_<^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)$$

Here $a=0$ & we know azimutal symetry then $m=0$

$$G(x,x') = 4\pi \sum_{l=0}^{\infty} P_l(\cos\theta') P_l(\cos\theta)\, r_<^l \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)$$

$$\Phi(x) = \frac{1}{4\pi\epsilon_0} \int d^3x\, \rho(x')\, G(x,x')$$

....